



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

$$(V_1) \begin{vmatrix} a_0 & a_0' + a_1 & a_1' + a_2 & a_2' + a_3 & a_3' + a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ \beta_0 & 2\beta_0' + \beta_1 & \beta_0'' + 2\beta_1' + \beta_2 & \beta_1'' + 2\beta_2' + \beta_3 & \beta_2'' + 2\beta_3' \\ 0 & \beta_0 & \beta_0' + \beta_1 & \beta_1' + \beta_2 & \beta_2' + \beta_3 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} \equiv 0,$$

$$(V_2) \begin{vmatrix} a_0 & a_0' + a_1 & a_1' + a_2 & a_2' + a_3 & a_4' \\ 0 & a_0 & a_1 & a_2 & a_3 \\ \beta_0 & 2\beta_0' + \beta_1 & \beta_0'' + 2\beta_1' + \beta_2 & \beta_1'' + 2\beta_2' + \beta_3 & \beta_3' \\ 0 & \beta_0 & \beta_0' + \beta_1 & \beta_1' + \beta_2 & \beta_2' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_3 \end{vmatrix} \equiv 0.$$

Hence V_1 and V_2 furnish the required necessary condition.

NEW YORK CITY, September, 1903.

LINEAR COVARIANTS OF THE BINARY QUADRATIC AND CUBIC.

By L. C. WALKER, Professor of Mathematics, Colorado School of Mines, Golden, Col.

The definition of *weight* is that every coefficient is of weight w measured by its suffix, and that every product of coefficients is of weight measured by the sum of the suffixes of its various factors.

A semi-covariant of the two quantics is a function of the two sets of coefficients, which is homogeneous in each set separately, and *isobaric* (equal weight) on the whole, though not necessarily in the sets separately.

The practice of speaking of a covariant whose dimensions are partial degrees i_1, i_2 in the two sets of coefficients and ω in the variables has of late become almost universal.*

The degrees of quantics in the variables are generally† spoken of as their orders p_1, p_2 .

The order ω , the partial degrees i_1, i_2 in the coefficients of the binary quadratic and cubic

$$(a_0, a_1, a_2)(x, y)^2, \quad (b_0, b_1, b_2, b_3)(x, y)^3,$$

and the weight w of the semi-invariant which is the leading coefficient C_0 in the linear covariant, are connected by the relation $i_1 p_1 + i_2 p_2 - \omega = 2w$.

Here $p_1 = 2, p_2 = 3, \omega = 1$, whence $2i_1 + 3i_2 - 1 = 2w$. More generally, if m be any positive integer and n any positive odd integer, we have, from the conditions of linear covariancy, $2mi_1 + 3ni_2 - 1 = 2w$. Thus the binary quadratic and cubic have an indefinite number of linear covariants.

*Elliott's *Algebra of Quantics*. †Ibid.

We now shall find the linear covariants,

(a) (2; 1, 2; 1, 3);	(b) (3; 2, 2; 1, 3);
(c) (5; 1, 2; 3, 3);	(d) (6; 2, 2; 3, 3).

(a). Assume for the semi-invariant the most general form

$$S \equiv C_0 \equiv a_2 b_2 + \lambda_1 a_1 b_1 + \lambda_2 a_2 b_0,$$

where λ_1, λ_2 are arbitrary multipliers. Operate on this with the two annihilators

$$\Omega_1 \equiv a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2}, \quad \Omega_2 \equiv b_0 \frac{d}{db_1} + 2b_1 \frac{d}{db_2} + 3b_2 \frac{d}{db_3},$$

and we obtain

$$a_0 b_1 [\lambda_1 + 2] + a_1 b_0 [2\lambda_2 + \lambda_1];$$

for which to vanish we must have

$$\lambda_1 + 2 = 0, \quad 2\lambda_2 + \lambda_1 = 0; \quad i.e., \lambda_1 = -2\lambda_2 = -2.$$

$$\therefore C_0 \equiv a_0 b_2 - 2a_1 b_1 + a_2 b_0,$$

$$C_1 \equiv \Omega_1 C_0 + \Omega_2 C_0 \equiv a_0 b_3 - 2a_1 b_2 + a_2 b_1,$$

where we have used the annihilators

$$O_1 \equiv 2a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1}, \quad O_2 \equiv 3b_1 \frac{d}{db_0} + 2b_2 \frac{d}{db_1} + b_3 \frac{d}{db_2}.$$

Thus the linear covariant is

$$I. \quad (a_0 b_2 - 2a_1 b_1 + a_2 b_0)x + (a_0 b_3 - 2a_1 b_2 + a_2 b_1)y.$$

The second transvectant of the quadratic and cubic gives I.

(b). Including all possible terms, the semi-invariant is of the form

$$S \equiv C_0 \equiv a_0^2 b_3 + \lambda_1 a_0 a_1 b_2 + \lambda_2 a_0 a_2 b_1 + \lambda_3 a_1^2 b_1 + \lambda_4 a_1 a_2 b_0.$$

Operate on this with Ω_1, Ω_2 . The vanishing of the expression requires four linear equations in the λ 's, from which we find

$$\lambda_1 = -3, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = -1.$$

$$\therefore S \equiv C_0 \equiv a_0^2 b_3 - 3a_0 a_1 b_2 + a_0 a_2 b_1 + 2a_1^2 b_1 - a_1 a_2 b_0,$$

$$C_1 \equiv O_1 C_0 + O_2 C_0 \equiv -(a_2^2 b_0 - 3a_1 a_2 b_1 + a_0 a_2 b_2 + 2a_1^2 b_2 - a_0 a_1 b_3).$$

The linear covariant is

$$\text{II. } C_0x + C_1y.$$

The first transvectant of I and the quadratic gives II. The third transvectant of the cubic and the square of the quadratic gives II.

(c). Assume the semi-invariant to be

$$\begin{aligned} S \equiv C_0 \equiv & a_0 b_0 b_2 b_3 + \lambda_1 a_0 b_1^2 b_3 + \lambda_2 a_0 b_1 b_2^2 + \lambda_3 a_1 b_0 b_1 b_3 \\ & + \lambda_4 a_1 b_0 b_2^2 + \lambda_5 a_1 b_1^2 b_2 + \lambda_6 a_2 b_0^2 b_3 + \lambda_7 a_2 b_0 b_1 b_2 + \lambda_8 a_2 b_1^2. \end{aligned}$$

Solving as in (a), we obtain eight linear equations in the λ 's, from which we find

$$\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = -4, \lambda_5 = 2, \lambda_6 = -1, \lambda_7 = 3, \lambda_8 = -2.$$

Thus the linear covariant is

$$\begin{aligned} \text{III. } & (a_0 b_0 b_2 b_3 - 2a_0 b_1^2 b_3 + a_0 b_1 b_2^2 + 2a_1 b_0 b_1 b_3 - 4a_1 b_0 b_2^2 \\ & + 2a_1 b_1^2 b_2 - a_2 b_0^2 b_3 + 3a_2 b_0 b_1 b_2 - 2a_2 b_1^3) x \\ & - (a_2 b_0 b_1 b_3 - 2a_2 b_0 b_2^2 + a_2 b_1^2 b_2 + 2a_1 b_0 b_2 b_3 - 4a_1 b_1^2 b_3 \\ & + 2a_1 b_1 b_2^2 - a_0 b_0 b_3^2 + 3a_0 b_1 b_2 b_3 - 2a_0 b_2^3) y. \end{aligned}$$

The first transvectant of I and the *Hessian* of the cubic gives III. The second transvectant of the quadratic and the *cubicovariant* of the cubic gives III.

(d). Here we assume the semi-invariant to be

$$\begin{aligned} S \equiv C_0 \equiv & a_0^2 b_0 b_3^2 + \lambda_1 a_0^2 b_2^3 + \lambda_2 a_0^2 b_1 b_2 b_3 + \lambda_3 a_0 a_1 b_0 b_2 b_3 + \lambda_4 a_0 a_1 b_1^2 b_3 \\ & + \lambda_5 a_0 a_1 b_1 b_2^2 + \lambda_6 a_0 a_2 b_0 b_1 b_3 + \lambda_7 a_0 a_2 b_0 b_2^2 + \lambda_8 a_0 a_2 b_1^2 b_2 \\ & + \lambda_9 a_1^2 b_0 b_1 b_3 + \lambda_{10} a_1^2 b_0 b_2^2 + \lambda_{11} a_1^2 b_1^2 b_2 + \lambda_{12} a_1 a_2 b_0^2 b_3 \\ & + \lambda_{13} a_1 a_2 b_0 b_1 b_2 + \lambda_{14} a_1 a_2 b_1^3 + \lambda_{15} a_2^2 b_0 b_2 + \lambda_{16} a_2^2 b_0 b_1^2, \end{aligned}$$

which includes all possible terms. As in (a), we obtain

$$\begin{aligned} & a_0^2 b_1^2 b_3 (\lambda_4 + 2\lambda_1) + a_0 a_2 b_0^2 b_3 (\lambda_{12} + \lambda_6) + a_0 a_2 b_1^3 (\lambda_{14} + 2\lambda_8) + a_1^2 b_0^2 b_3 (2\lambda_{12} + \lambda_9) \\ & + a_1^2 b_1^3 (2\lambda_{14} + 2\lambda_{11}) + a_2^2 b_0^2 b_1 (4\lambda_{16} + 4\lambda_{15}) + a_0^2 b_0 b_2 b_3 (\lambda_3 + \lambda_1 + 6) + a_0^2 b_1 b_2^2 \\ & (\lambda_5 + 6\lambda_2 + 3\lambda_1) + a_1 a_2 b_0^2 b_2 (4\lambda_{15} + \lambda_{12} + 3\lambda_{12}) + a_1 a_2 b_0 b_1^2 (4\lambda_{16} + 3\lambda_{14} + 2\lambda_{13}) + \\ & a_0 a_1 b_0 b_1 b_3 (2\lambda_9 + 2\lambda_8 + 2\lambda_4 + 2\lambda_3) + a_0 a_1 b_0 b_2^2 (2\lambda_{10} + 2\lambda_7 + \lambda_5 + 3\lambda_6) + a_0 a_1 b_1^2 b_2 \\ & (2\lambda_{11} + 2\lambda_8 + 4\lambda_5 + 3\lambda_4) + a_0 a_2 b_0 b_1 b_2 (\lambda_{13} + 2\lambda_8 + 4\lambda_7 + 3\lambda_6) + a_1^2 b_0 b_1 b_2 (2\lambda_{13} + 2\lambda_{11} \\ & + 4\lambda_{10} + 3\lambda_9). \end{aligned}$$

Its vanishing gives (5; 2, 2; 3, 3) relations which have to be satisfied by the (6; 2, 2; 3, 3) multipliers. If then (6; 2, 2; 3, 3) exceeds (5; 2, 2; 3, 3) we

can satisfy them, the number of the multipliers still arbitrary being $(6; 2, 2; 3, 3) - (5; 2, 2; 3, 3) \equiv 17 - 15 = 2$.

First, the first transvectant (1), of the cubic and the square of I; and (2), of the *Hessian* of the cubic and II: or, *second*, the second transvectant (1), of the cubic and the square of I; (2), of the quadratic and the product of I by the *Hessian* of the cubic; and (3), of the *Hessian* of the cubic and the product of the quadratic by I: or, *third*, the third transvectant of the *cubicovariant* of the cubic and the square of the quadratic—either *first* or *second* or *third* shows that the partitions* $a_2^2 b_0 b_2$ and $a_2^2 b_0 b_1^2$ are absent from this linear covariant. We then have $\lambda_{16} = \lambda_{15} = 0$. Now from the other fourteen linear equations in the λ 's we obtain

$$\begin{aligned}\lambda_1 &= 2, \quad \lambda_2 = \lambda_3 = -3, \quad \lambda_4 = 6, \quad \lambda_5 = -3, \quad \lambda_6 = -1, \quad \lambda_7 = 2, \\ \lambda_8 &= -1, \quad \lambda_9 = -2, \quad \lambda_{10} = 4, \quad \lambda_{11} = -2, \quad \lambda_{12} = 1, \quad \lambda_{13} = -3, \quad \lambda_{14} = 2.\end{aligned}$$

The required linear covariant is

$$\begin{aligned}&(a_0^2 b_0 b_3 - 3a_0^2 b_1 b_2 b_3 + 2a_0^2 b_2^3 - 3a_0 a_1 b_0 b_2 b_3 + 6a_0 a_1 b_1^2 b_3 - 3a_0 a_1 b_1 b_2^2 \\&- a_0 a_2 b_0 b_1 b_3 + a_0 a_2 b_0 b_2^2 - a_0 a_2 b_1^2 b_2 - 2a_1^2 b_0 b_1 b_3 + 4a_1^2 b_0 b_2^2 - 2a_1^2 b_1^2 b_2 \\&+ a_1 a_2 b_0^2 b_3 - 3a_1 a_2 b_0 b_1 b_2 + 2a_1 a_2 b_1^3) x + (a_0 a_1 b_0 b_2^2 - 3a_0 a_1 b_1 b_2 b_3 + 2a_0 a_1 b_2^3 \\&- a_0 a_2 b_0 b_2 b_3 + 2a_0 a_2 b_1^2 b_2 - a_0 a_2 b_1 b_2^2 - 2a_1^2 b_0 b_2 b_3 + 4a_1^2 b_1^2 b_3 - 2a_1^2 b_1 b_2^2 \\&- 3a_1 a_2 b_0 b_1 b_3 + 6a_1 a_2 b_0 b_1^2 - 3a_1 a_2 b_1^2 b_2 + a_2^2 b_0^2 b_3 - 3a_2^2 b_0 b_1 b_2 + 2a_2^2 b_1^3) y.\end{aligned}$$

The number of linearly independent semi-invariants of given weight w and partial degrees i_1, i_2 of the quadratic and cubic, is given by

$$(w; i_1, p_1; i_2, p_2) - (w-1; i_1, p_1; i_2, p_2).$$

This expression is not applicable when it exceeds unity for all values of m and n that do not give linear semi-invariants, because to each of these values corresponds one and only one linear covariant for a transvection of some combination (1), of the two quantics; or (2), of their covariants; or (3), of the quantics and their covariants. Professor Paul Gordan has proved that a complete system of transvectants is coextensive with a complete system of covariants, also including invariants as a particular case. For the geometrical interpretation of this system of quantics, see Art. 198 of Salmon's *Higher Algebra*.

* For the theory of numbers of partitions, see Professor Cayley's second memoir on Quantics (*Collected Works*, Vol. II).